Aperiodic Subshifts on Polycyclic Groups

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Aperiodic Subshifts on Nilpotent and Polycyclic Groups

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18th August 2016

Abstract

We prove that every polycyclic group of nonlinear growth admits a strongly aperiodic SFT and has an undecidable domino problem. This answers a question of [5] and generalizes the result of [2].

Subshifts of finite type (SFT for short) in a group G are colorings of the elements of G subject to local constraints. They have been studied extensively for the free abelian group $\mathbb{Z}$ [14] and on the free abelian group $\mathbb{Z}^2$, where they correspond to tilings of the discrete plane [13].

One of the main question about SFTs is the existence of strongly aperiodic SFT, that is finding a finite set of local constraints so that the only way to color the group G is to do so aperiodically. Strongly aperiodic SFT are known to exists for the group $\mathbb{Z}^2$ [4] and the author [10] has provided examples on many groups of the form $\mathbb{Z} \times G$.

A related question is the decidability of the domino problem: Given an SFT X on a group G, decide if X is empty. This question is somewhat related to the existence of *weakly* aperiodic SFT, i.e. where no coloring is periodic along a subgroup of finite index in G. The undecidability of the domino problem has been established on $\mathbb{Z}^2$ [4], on Baumslag-Solitar groups [1] and on any nilpotent group of nonlinear growth [2].

In this article we introduce a few elementary techniques that show how to exhibit strongly aperiodic SFT on any polycyclic group which is not of linear growth (i.e. not virtually $\mathbb{Z}$), and we also prove that any such group has an undecidable domino problem. The first result answers a question of Carroll and Penland [5], while the second generalizes the result of Ballier and Stein from nilpotent groups to polycyclic groups.

A full characterization of groups (or even finitely presented groups) with strongly aperiodic SFT is still for now out of reach.

1 Definitions

We assume some familiarity with group theory and actions of groups. See [6] for a good reference on symbolic dynamics on groups. All groups below are implicitely supposed to be finitely generated (f.g. for short).

Let A be a finite set and G a group. We denote by A^G the set of all functions from G to A. For $x \in A^G$, we write x_g instead of $x(g)$ for the value of x in g.

G acts on A^G by

$$(g \cdot x)_h = x_{g^{-1}h}$$

A *pattern* is a partial function P of G to A with finite support. The support of P will be denoted by $\mathrm{Supp}(P)$.

A *subshift* of A^G is a subset X of A^G which is topologically closed (for the product topology on A^G) and invariant under the action of G.

A subshift can also be defined in terms of forbidden patterns. If $\mathcal{P}$ is a collection of patterns, the subshift defined by P is

$$X_{\mathcal{P}} = \left\{x \in A^G | \forall g \in G, \forall P \in \mathcal{P} \exists h \in \mathrm{Supp}(P), (g \cdot x)_h \neq P_h\right\}$$

Every such set is a subshift, and every subshift can be defined this way. If X can be defined by a finite set $\mathcal{P}$, X is said to be a *subshift of finite type*, or for short a SFT.

For a point $x \in X$, the stabilizer of x is $Stab(x) = \{g | g \cdot x = x\}$

A subshift X is *strongly aperiodic* if it is nonempty and every point of X has a finite stabilizer. Some authors require that the stabilizer of each point is trivial (rather than finite), this will not make any difference in this article.

A f.g. group G is said to have *decidable domino problem* if there is an algorithm that, given a description of a finite set of patterns $\mathcal{P}$, decides if $X_{\mathcal{P}}$ is empty.

In the remaining, we are interested in groups G which admit strongly aperiodic SFTs or have an undecidable domino problem.

We now summarize previous theorems:

Theorem 1.
- *$\mathbb{Z}$ does not admit strongly aperiodic SFT and has decidable word problem*
- *Free groups have decidable domino problem [12]*
- *The free abelian group $\mathbb{Z}^2$ [4] has a strongly aperiodic SFT and an undecidable domino problem*
- *The free abelian group $\mathbb{Z}^3$ [9] has a strongly aperiodic SFT.*
- *f.g. nilpotent groups have an undecidable word problem unless they are virtually cyclic [2]*

We also give a few structural results:

Theorem 2.

- *Let G, H be f. g. commensurable groups. Then G admits a strongly aperiodic SFT (resp. has an undecidable domino problem) if only if H does. [5]*
- *Let G, H be* finitely presented *groups that are quasi-isometric. Then G admits a weakly aperiodic SFT (resp. has an undecidable domino problem) if and only if H does. [7]*
- *Finitely presented groups with a strongly aperiodic SFT have decidable word problem [10].*

The first of these results will be used almost everywhere in what follows.

2 Building aperiodic SFTs on G from aperiodic SFTs on subgroups and quotients of G

2.1 Building blocks

We start by a few easy propositions that explain how to build SFT in G from a SFT in a subgroup H or a SFT in G/H, provided in the second case that H is normal and finitely generated.

Proposition 2.1. *Let G be a f.g. group and H a f.g. normal subgroup of G. Let $\phi : G \to G/H$ the corresponding morphism. Let X be a SFT on G/H over the alphabet A. Let*

$$Y = \{y \in A^G | \exists x \in X \forall g \in G, y_g = x_{\phi(g)}\}$$

Then Y is a SFT.

Furthermore, for every $y \in Y$, there exists $x \in X$ s.t. $Stab(y) = HStab(x)$.

See e.g. [5] for a proof. The idea is to lift the forbidden patterns defined on G/H to forbidden patterns defined on G, and to use additional forbidden patterns to force $y_g = y_{g'}$ whenever $gg'^{-1} \in H$. This is possible as H is supposed to be finitely generated.

Proposition 2.2. *Let G be a f.g. group and H a f.g. subgroup of G.*

Let X be a SFT on H over the alphabet A defined by the set of forbidden patterns $\mathcal{P}$. Using the same forbidden patterns, we obtain a SFT on G that we call Y.

Then Y is nonempty iff X is nonempty. More precisely, let K be a left transversal of H in G. Then $y \in Y$ iff there exists points $(x^k)_{k\in K}$ in X s.t. $y_{kh} = x^k_h$.

In particular, for all $y \in Y$, there exists $x \in X$ s.t. $Stab(y) \cap H \subseteq Stab(x)$

Proof. Write $G = KH$ for some transversal K. Each element of $g \in G$ can therefore be written in a unique way in the form $g = kh$.

Let $y \in Y$ and let $k \in K$. Define $(x^k)_h = y_{kh}$. Let $h' \in H$ and P a forbidden pattern of $\mathcal{P}$. We will prove there exists p s.t. $(h' \cdot x^k)_p \neq P_p$ which proves that $x^k \in X$.

By definition of Y, there exists p in the support of $\mathcal{P}$ s.t. $(h'k^{-1} \cdot y)_p \neq P_p$. Therefore $y_{kh'^{-1}p} \neq P_p$. But $(h' \cdot x^k)_p = (x^k)_{h'^{-1}p} = y_{kh'^{-1}p}$, the result is proven.

Conversely, take some points $(x^k)_{k\in K}$ in X and define $y_{kh} = x^k_h$. Let $g \in G$. Write $g^{-1} = kh^{-1}$ for some k and h. Let $P \in \mathcal{P}$. As $x^k \in X$, there exists $p \in H$ s.t. $(h \cdot x^k)_p \neq P_p$. But then $(g \cdot y)_p = y_{g^{-1}p} = y_{kh^{-1}p} = x^k_{h^{-1}p} = (h \cdot x^k)_p \neq P_p$. Therefore $y \in Y$. □

2.2 Applications

We now start with the first proposition that gives a natural way to prove that a group G has a strongly aperiodic SFT.

Proposition 2.3. *Let G be a f.g. group. Suppose that G contains two f.g. groups H_1, H_2 s.t.*

- *$H_1 \subseteq H_2$.*
- *H_1 is normal and G/H_1 admits a strongly aperiodic SFT X_1*
- *H_2 admits a strongly aperiodic SFT X_2*

Then G admits a strongly aperiodic SFT Y. Furthermore, if each point in X_1 and X_2 have trivial stabilizers, then every point of Y has a trivial stabilizer.

It is a good but nontrivial exercise to show, under the conditions of the proposition, that if H_1 and H_2 have a decidable word problem, then G does.

Proof. Use the two previous propositions to build Y_1 and Y_2 and consider $Y = Y_1 \times Y_2$.

Let $y = (y_1, y_2) \in Y$ and consider its stabilizer $K = Stab(y)$. By definition of Y_1, $K \subseteq H_1F$ for some finite set F. By definition of Y_2, $K \cap H_2$ is finite. In particular $K \cap H_1$ is finite.

This implies that K is finite. Indeed, for each $f \in F$, $K \cap fH_1 = K \cap H_1 f$ is finite: If $x, y \in K \cap fH_1$, then $xy^{-1} \in K \cap H_1$.

Furthermore if F is trivial and $K \cap H_2$ is trivial, then K is trivial. □

Corollary 2.4. *Let G be a f.g. group. If G contains a f.g. normal subgroup H_1 s.t. G/H_1 admits a strongly aperiodic SFT and H_1 admits a strongly aperiodic SFT, then G admits a strongty aperiodic SFT.*

In particular, if G_1 and G_2 are f.g. groups that admit strongly aperiodic SFTs, then $G_1 \times G_2$ does.

Therefore groups with strongly aperiodic SFT are closed under direct product. This seems quite natural but does not seem to have been known.

The second proposition of the previous section explain how an aperiodic SFT on $H \subseteq G$ give rise to a SFT on G. Usually this SFT will not be aperiodic. There are however a few conditions in which it will. Here is an obvious one, for which we do not have any interesting application

Proposition 2.5. *Let H be a subgroup of G that admits an aperiodic SFT X. Suppose that every element of G is conjugate to some element of H (H is said to be* conjugately-dense*). Then G admits an aperiodic SFT. Actually, X, seen as an aperiodic SFT on G, is aperiodic.*

3 Aperiodic SFTs on polycyclic and nilpotent groups

Using the previous proposition, we will be able to prove that nontrivial nilpotent and polycyclic groups have strongly aperiodic SFT. We start with nilpotent groups.

First, a few definitions, see [15, Chapter 1] for details.

Let G be a group. The *Hirsch number* $h(G)$ of G is the number of infinite factors in a series with cyclic or finite factors. It is defined for nilpotent groups, and more generally for polycyclic groups. A nilpotent (or a polycyclic) group of Hirsch number 0 is a finite group. A nilpotent (or a polycyclic) group of Hirsch number 1 is a finite-by-cyclic-by-finite group, hence virtually $\mathbb{Z}$.

The only thing we will need about nilpotent groups is that (a) they have a non trivial center (b) quotients and subgroups of nilpotent groups are nilpotent and finitely generated (c) $h(G) = h(G/H) + h(H)$ (which make sense due to the previous point) (d) every nilpotent group contains a torsion-free nilpotent group of finite index.

Note that the rank and the Hirsch number coincide for free abelian groups, i.e. $\mathbb{Z}^n$ is of Hirsch number n.

Theorem 3. *Let G be a finitely generated group of nonlinear polynomial growth. Then G admits a strongly aperiodic SFT.*

Proof. Such groups are exactly the f.g. virtually nilpotent groups of Hirsch number different from 1. The proof will use repeatedly that if G is a f.g. group, and H a subgroup of finite index, then G has a strongly aperiodic SFT iff H does, see [5] for details. In particular it is sufficient to prove the theorem for f.g. nilpotent groups to obtain the results for f.g. virtually nilpotent groups.

The result is clearly true for Hirsch number 0. We now prove the theorem by induction on the Hirsch number $h(G) \geq 2$.

We start with Hirsch number 2. A nilpotent group of Hirsch number 2 is virtually $\mathbb{Z}^2$, hence has a strongly aperiodic SFT.

Now let G be a nilpotent group of Hirsch number $n > 2$. Let G_1 be a torsion-free nilpotent subgroup of finite index of G. It is enough to prove the theorem for G_1 to obtain the theorem for G, therefore we will suppose that $G_1 = G$.

G has a non trivial center and torsion-free, hence contains a copy of $H_1 = \mathbb{Z}$ in its center. $h(G/H_1) = h(G) - h(H_1) \geq 2$ therefore by induction G/H_1 has a strongly aperiodic SFT.

Furthermore, G/H_1 is a nilpotent group, hence contains a torsion-free nilpotent group of finite index, hence contains a torsion-free element x. Let y be a representative of x in G. Let H_2 be the group generated by H_1 and y. AS H_1 is normal in H_2, $h(H_2) = h(H_2/H_1) + h(H_1) = 2$ Therefore H_2 is by construction of Hirsch length 2, hence has a strongly aperiodic SFT.

We have produced our required groups H_1 and H_2 and we can apply Proposition 2.3 to finish the induction. □

We now proceed to the proof for a polycyclic group. The proof is roughly similar, except there is an additional case to work out, which is the case of groups G that admit an exact sequence $1 \to \mathbb{Z}^n \to G \to \mathbb{Z} \to 1$. It is not obvious how to obtain strongly aperiodic SFTs for these groups. Fortunately:

Theorem 4 ([3]). *Let G be a group that admits an exact sequence*

$$1 \to \mathbb{Z}^n \to G \to \mathbb{Z} \to 1$$

with $n \geq 2$.

Then G admits a strongly aperiodic SFT.

This result is highly nontrivial and $\mathbb{Z}$ can actually be replaced by any f.g. group with decidable word problem. We prefer citing the result in this form as it is likely it admits a simpler proof in this particular case.

Polycyclic groups are groups which admit a series with cyclic factors. All relevant properties of nilpotent groups stated above are still true when nilpotent is replaced by polycyclic: polycyclic groups are always finitely generated (actually finitely presented), quotients and subgroups of polycyclic groups are polycyclic, every polycyclic group contains a torsion-free polycyclic group of finite index. Moreover polycyclic groups admit a nontrivial normal free abelian subgroup.

Theorem 5. *Let G be a virtually polycyclic group of Hirsch number different from 1.*

Then G admits a strongly aperiodic SFT.

Proof. We start with Hirsch number 2. In this case, the polycyclic group G is virtually $\mathbb{Z}^2$, hence has a strongly aperiodic SFT.

For the induction, consider G a polycyclic group of Hirsch number $n > 2$. G contains a torsion-free polycyclic group of finite index so we may suppose as before that G is torsion free.

G contains an nontrivial normal free abelian subgroup $H_1 = \mathbb{Z}^k$.

There are four cases:

- $k = n$. In this case, $h(G/H_1) = h(G) - h(H_1) = 0$, therefore G is virtually $\mathbb{Z}^n$ and admits a strongly aperiodic SFT.
- $k = 1$. We use the induction hypothesis on G/H_1 of Hirsch number at least $n-1$. By taking any element of infinite order in G/H_1, we obtain, as in the nilpotent case, a group $H_2 \supset H_1$ of Hirsch number 2 and we apply Proposition 2.3.
- $1 < k < n-1$. We use the induction hypothesis on both G/H_1 and $H_2 = H_1$ and conclude by Proposition 2.3.
- $k = n-1$. In this case G/H_1 is of Hirsch number 1, hence virtually $\mathbb{Z}$.

  By taking a finite index subgroup G_1 of G we can suppose wlog that G/H_1 is exactly $\mathbb{Z}$. We can then apply the theorem of [3].

□

Note that the full extent of the theorem in [3] covers all cases except the case $k \leq 1$. The reason we stated it only for the case $k = n-1$ is that we believe there is an easier proof in this case.

4 The domino problem

We now prove that the domino problem is undecidable for all virtually polycyclic groups which are not virtually cyclic (i.e. which are of Hirsch number greater than 2).

Proposition 4.1. *Let G be a virtually polyclic group of Hirsch number greater than 2. Then G admits a f.g. subgroup H that factors onto $\mathbb{Z}^2$.*

Proof. By induction on the Hirsch number, it is sufficient to prove the result for polycyclic groups. Every polycyclic group of Hirsch number 2 is virtually $\mathbb{Z}^2$, there is nothing to prove.

Let G be a polycyclic group of Hirsch number at least 3 and let N be a nontrivial normal free abelian subgroup of G. If N is of rank greater than 2, G contains a subgroup isomorphic to $\mathbb{Z}^2$, there is nothing to prove.

Otherwise, G/N is a polycyclic group of Hirsch number at least 2, and admits by induction a subgroup H that factors onto $\mathbb{Z}^2$. Therefore G admits a subgroup (the preimage of H) that factors onto $\mathbb{Z}^2$. □

Corollary 4.2. *Every virtually polycyclic group which is not virtually cyclic has an undecidable domino problem.*

Proof. We use the previous proposition to obtain H. As G is polycyclic, H is polycyclic. Then $H/N = \mathbb{Z}^2$ for a normal subgroup N of H, which is finitely generated as G (therefore H) is polycyclic.

Therefore H has an undecidable domino problem: Given a SFT X on H/N, we may obtain (constructively) a SFT Y on H s.t. X is empty iff Y is empty by Proposition 2.1.

Therefore G has an undecidable domino problem: Given a SFT X on H, we may obtain (constructively) a SFT Y on G s.t. X is empty iff Y is empty by Proposition 2.2. □

5 Conclusion

Proposition 2.2 gives a way to transform a SFT X on H to a SFT Y on $G \supseteq H$. In many cases, Y will not be strongly aperiodic. However, surprisingly, it is the case in some situations. We already have used this fact in the proof of our main theorem. Another example is that $\mathbb{Z}[1/2] \times \mathbb{Z}[1/2]$ has a strongly aperiodic SFT Y , which is basically obtained from any strongly aperiodic SFT X on $\mathbb{Z}^2$. The idea is that if $g \in \mathbb{Z}[1/2] \times \mathbb{Z}[1/2]$ is nontrivial, then some power of g is in $\mathbb{Z} \times \mathbb{Z}$, therefore no nontrivial element of g can be stabilizer of some point of Y.

Of course $\mathbb{Z}[1/2] \times \mathbb{Z}[1/2]$ is an infinitely generated group, and the study of SFT is primarily interesting in finitely generated groups. However this observation might be useful for example to produce a strongly aperiodic SFT in some solvable groups, e.g. $BS(1,2) \times BS(1,2)$ ($BS(1,2)$ is basically a semidirect product of $\mathbb{Z}[1/2]$ and $\mathbb{Z}$).

An obvious natural generalization of the main theorem would be to deal with solvable groups rather than polycyclic groups. The difficulty is that solvable groups that are not polycyclic always have abelian subgroups which are not finitely generated. This adds additional cases that we do not know how to treat. Of particular interest is the lamplighter group, or solvable Baumslag-Solitar groups (for which only weakly aperiodic SFTs are known). Furthermore, it is not true that every solvable group has a strongly aperiodic SFT as there exist solvable finitely presented group with an undecidable word problem [8].

Another possible generalization which is promising is Noetherian groups, which are groups where every subgroup is finitely generated. The wilder examples of Noetherian groups, Tarski monsters, do admit strongly aperiodic SFT X [10] (where points in X might have finite, non trivial stabilizer), so it is quite possible that all Noetherian groups (which are not virtually cyclic) do admit strongly aperiodic SFT.

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A Automorphism-free SFT

We finish with a new definition of aperiodicity

Definition A.1. *Let G be a group. Let $Aut(G)$ be the group of automorphisms of G. For $\phi \in Aut(G)$, let $\phi(x)$ be the point defined by $\phi(x)_g = x_{\phi(g)}$.*

For a subshift X and a point x, let $Div(x, X) = \{\phi \in Aut(G), \phi(x) \in X\}$

We say that a nonempty subshift X is automorphism-free if $Div(x, X)$ is trivial for all $x \in X$.

The interest of automorphism-free SFT is seen by the following remarks:

Proposition A.2. *Let $G = \mathbb{Z}^n$ with $n > 1$. Then an automorphism-free SFT X of G is strongly aperiodic.*

Proof. In this proof, we will see $\mathbb{Z}^n$ as a $\mathbb{Z}$-module and will write the law group additively and not multiplicatively.

Let $x \in X$. Let $u \in \mathbb{Z}^n$. Let v be any nonzero element of $\mathbb{Z}^n$ that is orthogonal to u, that is $uv^T = 0$.

Define $\phi(g) = g + (gv^T)u$. Then ϕ is an automorphism of $\mathbb{Z}^n$, the inverse being given by $\psi(g) = g - (gv^T)u$. (In a basis with some basis vectors collinear to u and v, ϕ would be a identity matrix with one other nonzero coefficient).

Suppose that u is a vector of periodicity for x, i.e. $u \cdot x = x$, therefore $x_{g-u} = x_g$ for all $g \in \mathbb{Z}^n$

However, for all $g \in \mathbb{Z}^n$, gv^T is an integer, therefore $\phi(g) = g + ku$ for some k depending on g. Therefore $x_{\phi(g)} = x_g$.

In particular $\phi(x) = x$. Therefore $\phi(x) \in X$. Therefore ϕ is trivial, that is $u = 0$. ☐

And the fact that automorphism-free SFT do exist:

Proposition A.3. *Let $n > 1$. Then $\mathbb{Z}^n$ admits a automorphism-free SFT.*

Proof. Let $\{e_1, e_2 \ldots e_n\}$ be the canonical base of $\mathbb{Z}^n$. Let V_i be the module generated by all vectors $\{e_j, j \neq i\}$.

We will first build, for all i, a SFT X_i s.t. if $x \in X_i$ and ϕ is an automorphism of $\mathbb{Z}^n$ s.t. $\phi(x) \in X_i$, then $\phi(V_i) = V_i$. It is sufficient to do it for $i = 1$.

To do this, we will start from a SFT Y on $A^{\mathbb{Z}^2}$ built by Kari [11]. On this SFT, a mapping $\pi : A \mapsto \{0, 1\}$ can be defined s.t. for any configuration $x \in Y$, every column of $\pi(x)$ (i.e. in direction e_2) is monochromatic, and every row contains a sturmian word of irrational slope. This implies in particular that every line of the form $(kpe_1)_{k\in\mathbb{Z}}$, with $p \neq 0$, cannot be monochromatic.

We extend this SFT Y to a SFT X_1 on $\mathbb{Z}^n$ by imposing every configuration to be periodic in direction $e_3 \ldots e_n$. Doing this, we have built a (nonempty) SFT with the following property: for every point $x \in X_1$:

- $\pi(x_{p_1e_1+p_2e_2+\ldots p_ne_n}) = \pi(x_{p_1e_1})$
- $(\pi(x_{p_1ke_1}))_{k\in\mathbb{Z}}$ is not monochromatic unless $p_1 = 0$.

Now, let $x \in X_1$ and ϕ an automorphism of $\mathbb{Z}^n$ s.t. $y = \phi(x) \in X_1$. Let $\phi(e_2) = p_1e_1 + p_2e_2 + \dots p_ne_n$. As $y \in X_1$, $(\pi(y_{ke_2}))_{k\in\mathbb{Z}}$ must be monochromatic. But $\pi(y_{ke_2}) = \pi(\phi(x)_{ke_2}) = \pi(x_{\phi(ke_2)}) = \pi(x_{kp_1e_1})$. Therefore $p_1 = 0$. Doing the same with $e_3 \dots e_n$, we have proven that if $\phi(x) \in X_1$, then $\phi(V_1) = V_1$.

We build in the same way SFTs $X_2, X_3 \dots X_n$ with similar properties and we take $X = X_1 \times X_2 \dots \times X_n$.

By the previous discussion, if $x \in X$ and $\phi(x) \in X$ then $\phi(V_i) = V_i$ for all i. This implies in particular $\phi(e_i) = \pm e_i$.

To finish, let Z be the SFT over the alphabet $\{0, 1, 2\}^n$ that consists in the point z defined by $z_{p_1e_1+\dots p_ne_n} = (p_1 \mod 3, p_2 \mod 3, \dots p_n \mod 3)$ and its $3^n - 1$ other translates.

It is easy to see that if $z \in Z$ and $\phi(z) \in Z$ then it is not possible to have $\phi(e_i) = -e_i$ for some i.

Therefore $X \times Z$ is automorphism-free. $\square$